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GLOBAL ATTRACTORS FOR A CLASS NONLINEAR EVOLUTION EQUATION

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Abstract

In this paper, using a new method (or framework), we deal with the dissipative feature of the solution semigroup. And then we establish the existence of global attractor for a class nonlinear evolution equation in $H_0^1(\Omega) \times H_0^1(\Omega)$, where the nonlinear term f satisfies a critical exponential growth condition.

1. Introduction

In this paper, we study the long-time behaviors of the solutions for the following nonlinear evolution equation:

$$\begin{cases} (|u_t|u_t)_t - \Delta u - \mu\Delta u_t - \Delta u_{tt} = f(u) & \text{in} \quad \Omega, \\ u|_{t=0} = u_0,\ u_t|_{t=0} = u_1 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is an open bounded set of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, where $\mu > 0$. Equation (1.1), which appear as a class nonlinear evolution equation, like Karman equation, is used to represent the flow of condensability airs in the across velocity of sound district, see [7]. For the class nonlinear evolution equation, the existence of global solutions has been studied in [2, 7] etc. In [7], authors have discussed the existence of global solutions in $H_0^1(\Omega) \times H_0^1(\Omega)$. However, as we know, the long-time behaviors of solutions of (1.1) have not been considered up to now. In this paper, we try to discuss the problem.

The key idea of the paper is to use a new method (or framework), we deal with the dissipative feature of the solution semigroup $\{S(t)\}_{t\geq 0}$. And then by verifying the convenient condition, i.e., asymptotically smooth introduced in [1, 3, 5], we obtain the necessary compactness for the semigroup $\{S(t)\}_{t\geq 0}$ in $H_0^1(\Omega) \times H_0^1(\Omega)$. So we obtain global attractors of the product space.

2. Functional Setting

In what follows, we give some notations which will be used throughout this paper. Let Ω be a bounded subset of $\mathbb{R}^n$ with a sufficiently smooth boundary, $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$ corresponding norms $\|u\| = (\int_\Omega |\nabla u|^2)^{\frac{1}{2}}$, and $|u| = (\int_\Omega |u|^2)^{\frac{1}{2}}$, respectively, the norms in $L^p(\Omega)(3 \leq p < \infty)$ are denoted by $|u|_p = (\int_\Omega |u|^p)^{\frac{1}{p}}$. The product Hilbert spaces

$$E_0 = V \times V.$$

For any $(u, v) \in E_0$ the norm in E_0 is denoted by

$$\| (u, v) \|_0^2 = \|u\|^2 + \|v\|^2. \tag{2.1}$$

Denote by C any positive constant which may be different from line to line even in the same line.

Lemma 2.1 [6]. *Let* Φ *be an absolutely continuous positive function on* $\mathbb{R}^+$, *which satisfies for some* $\varepsilon > 0$ *the differential inequality*:

$$\frac{d}{dt}\Phi(t) + 2\varepsilon\Phi(t) \leq g(t)\Phi(t) + h(t), \tag{2.2}$$

for almost every $t \in \mathbb{R}^+$, *where g and h are functions on* $\mathbb{R}^+$ *such that there exists* $t_0 \geq 0$

$$\int_\tau^t |g(y)|dy \leq \alpha_1(1 + (t - \tau)^\mu), \qquad \forall t_0 \geq t \geq \tau \geq 0,$$

for some $\alpha_1 \geq 0$ *and* $\mu \in [0, 1)$, *and*

$$\sup_{t_0 \geq t \geq 0} \int_t^{t+1} |h(y)|dy \leq \beta_1,$$

for some $\beta_1 \geq 0$.

$$\int_\tau^t |g(y)|dy \leq \alpha_2(1 + (t - \tau)^\mu), \qquad \forall t \geq \tau \geq t_0,$$

for some $\alpha_2 \geq 0$ *and* $\mu \in [0, 1)$, *and for some* $\beta_2 \geq 0$

$$\sup_{t \geq t_0} \int_t^{t+1} |h(y)|dy \leq \beta_2,$$

where α_2 *and* β_2 *are constants and independent of t and* t_0.

Then there exist constants γ_1, γ *only dependent of* α_1, α_2 *and* ε *such that*

$$\Phi(t) \leq (\gamma\Phi(0) + \gamma_1\beta_1 C(\varepsilon, t_0))e^{-\varepsilon t} + \rho, \qquad \forall t \in \mathbb{R}^+,$$

holds, where

$$C(\varepsilon, t_0) = \frac{e^\varepsilon}{1 - e^{-\varepsilon}} e^{\varepsilon t_0}, \qquad \rho = \frac{\alpha_2\beta_2 e^\varepsilon}{1 - e^{-\varepsilon}}$$

are positive constants.

For the proof and more details, we refer the reader to [6].

Remark. In the application of the above lemma, we might not have required regularity for Φ. However, this is not really a problem, since we can always suppose to work within a proper regularization scheme.

3. Global Attractor in $H_0^1(\Omega) \times H_0^1(\Omega)$

We are interested in the initial boundary value problem (1.1) involving a scalar function $u = u(x, t)$, where $u_0 = u_0(x)$ and $u_1 = u_1(x)$ are given $u_0, u_1 \in V$. The nonlinear term f grows with critical exponent. Throughout the paper we assume that function $f \in C(\mathbb{R}, \mathbb{R})$ satisfies the following conditions:

$$|f(r) - f(s)| \leq C|r - s|(1 + |r|^4 + |s|^4), \quad \forall r, s \in \mathbb{R}, \tag{3.3}$$

also, let f admit the decomposition

$$f = f_0 + f_1,$$

with $f_0, f_1 \in C(\mathbb{R})$, satisfying

$$|f_0(s)| \leq C(1 + |s|^5), \quad \forall s \in \mathbb{R}, \tag{3.4}$$

$$f_0(s)s \leq 0, \quad \forall s \in \mathbb{R}, \tag{3.5}$$

$$|f_1(s)| \leq C(1 + |s|^p), \quad p < 5, \forall s \in \mathbb{R}, \tag{3.6}$$

$$\limsup_{|s| \to \infty} \frac{f_1(s)}{s} < \lambda_1. \tag{3.7}$$

Without loss of generality, we can think p large enough, say, $p \geq 3$. Notice that by (3.7), there exists $\lambda < \lambda_1$, such that

$$f(s)s \leq f_1(s)s \leq \lambda s^2 + C. \tag{3.8}$$

We denote by F the function

$$F(u) = \int_{\Omega}\int_0^u f(s)ds,$$

which is easily seen to satisfy the inequalities

$$F(u) \leq \frac{1}{2}\lambda|u|^2 + C, \tag{3.9}$$

$$\langle f(u), u\rangle \leq F(u) + \frac{1}{2}\lambda|u|^2 + C. \tag{3.10}$$

By the standard Faedo-Galerkin methods, we can obtain a unique solution (u, u_t) of equations (1.1), here we only formulate the result:

Theorem 3.2 [7]**.** *The hypotheses are the general hypotheses above on* V, H *and we assume that f satisfies the hypotheses* (3.3) ~ (3.7), *and* $\mu > 0$ *satisfies* (1.2). *Let* $u_0, u_1,$ *and* $\mu > 0$ *be given, then there is a unique solution u of* (1.1) *satisfies*

$$u, \quad u_t \in L^\infty(0, T; V) \cap \mathcal{C}([0, T]; V),$$

$$u_{tt} \in L^\infty(0, T; V),$$

holds for any $T > 0$ *and* $(u_0, u_1) \in E_0$.

The initial boundary value problem is equivalent to continuous semigroup (see [7] etc.) $\{S(t)\}_{t\geq 0}$ defined by

$$S(t): E_0 \to E_0 \quad \text{and} \quad S(t): (u_0, u_1) \mapsto (u, u_t).$$

3.1. Bounded absorbing set

In this subsection, we assume that the nonlinearity *f* is a C^0 function and satisfies (3.3)~(3.7), and $\mu > 0$ is given and satisfies (1.2). Associated with (3.8), (3.9) and (3.10), we can prove the following results:

Lemma 3.3. *Assume that*

$$\|(u_0, u_1)\|_0 < R$$

for some $R > 0$. *Then we have the estimate*

$$|u_t(t)|_r^r + \int_0^t \| u_t(s)\|^2 ds \le \Lambda(R),$$

for any $t > 0$.

Proof. We take the scalar product in $L^2(\Omega)$ (1.1) with u_t and let

$$H(t) = \frac{2}{3}|u_t|_3^3 + \frac{1}{2}\|u\|^2 + \frac{1}{2}\|u_t\|^2 - F(u), \tag{3.11}$$

then we get

$$\frac{d}{dt}H(t) + \mu\|u_t\|^2 = 0, \tag{3.12}$$

and

$$H(t) + \mu\int_0^t \|u_t\|^2 = H(0), \tag{3.13}$$

according to Young's inequality and (3.9) we have

$$H(t) \ge \frac{1}{2}(|u_t|_3^3 + \|u_t\|^2) - M_{11},$$

and basis to (3.13)

$$|u_t(t)|_3^3 + \int_0^t \|u_t(s)\|^2 ds \le \Lambda(R), \tag{3.14}$$

where $C_\mu = \min\{1/2, \mu\}$ and $\Lambda(R) = \dfrac{H(0) + M_{11}}{C_\mu}$.

Theorem 3.4. $\{S(t)\}_{t\ge 0}$ *has a bounded absorbing set* B_0 *in* E_0, *that is, for any bounded subset* $B \subset E_0$, *there exists* $T_0 = T_0(B)$ *such that*

$$S(t)B \subset B_0 \quad \forall t \ge t_0.$$

Proof. We set $v = u_t + \delta u$ and rewrite the equation of (1.1) as follows:

$$(|u_t|u_t)_t - (\mu - \delta)\Delta v - \Delta v_t - (1 - \mu\delta + \delta^2)\Delta u = f(u), \tag{3.15}$$

and we take the inner product with v and set δ small enough and

$$E_1(t) = \frac{2}{3}|u_t|_3^3 + \delta\int_\Omega |u_t|^{r-2}u_t u + \frac{1}{2}\|v\|^2$$

$$+ \frac{1}{2}(1 - \mu\delta + \delta^2)\|u\|^2 - F(u), \tag{3.16}$$

$$E_2(t) = \delta(1 - \mu\delta + \delta^2)\|u\|^2 + (\mu - \delta)\|v\|^2 - \delta < f(u),\ u > +\delta|u_t|_3^3, \tag{3.17}$$

then we obtain

$$E_1(t) = -\int_0^t E_2(s)ds + 2\delta\int_0^t |u_t(s)|_3^3 ds + E_1(0). \tag{3.18}$$

The Young's inequality gives, there exist positive constants C_1, C_2, k_1 and k_2 only depend on Ω, δ such that

$$E_1(t) \geq C_1\left(\frac{2}{3}|u_t|_3^3 + \|v\|^2 + \varpi\|u\|^2\right) - \frac{C_1}{\varpi}|u_t|_3^4 - k_1; \tag{3.19}$$

and

$$E_2(t) \geq C_2\left(\frac{2}{3}|u_t|_3^3 + \|v\|^2 + \varpi\|u\|^2\right) - k_2, \tag{3.20}$$

here $\varpi = 1 - \dfrac{\lambda}{\lambda_1} - \mu\delta + \delta^2 > 0$ as δ small enough. By (3.14), this shows that

$$C_1\left(\frac{2}{3}|u_t|_3^3 + \|v\|^2 + \varpi\|u\|^2\right) - k_1$$

$$\leq -\int_0^t C_2\left(\frac{2}{3}|u_t|_3^3 + \|v\|^2 + \varpi\|u\|^2 - \frac{k_2}{C_2}\right)$$

$$+ 2\delta\int_0^t |u_t(s)|_3^3 ds + C_1\frac{1}{\varpi}|u_t|_3^4 + E_1(0)$$

$$\leq -\int_0^t C_2\left(\frac{2}{3}|u_t|_3^3 + \|v\|^2 + \varpi\|u\|^2 - \frac{k_2}{C_2}\right)$$

$$+\Lambda(R)^{\frac{2}{3}}\left(2\delta\int_0^t |u_t(s)|_3^2 ds + C_1 \frac{1}{\varpi}|u_t|_3^3\right) + E_1(0). \quad (3.21)$$

Then we get

$$C_1\left(\frac{2}{3}|u_t|_3^3 + \|v\|^2 + \varpi\|u\|^2\right) - k_1$$

$$\leq -\int_0^t C_2\left(\frac{2}{3}|u_t|_3^3 + \|v\|^2 + \varpi\|u\|^2 - \frac{k_2}{C_2}\right) + \Lambda_1(R) + E_1(0). \quad (3.22)$$

For any $M_1 > \frac{k_2}{C_2}$, there exists $t_1 = t_1(B)$ such that

$$\frac{2}{3}|u_t(t_1)|_3^3 + \|v(t_1)\|^2 + \varpi\|u(t_1)\|^2 \leq M_1.$$

According to (3.14), Lemma 3.3, it follows that if $t \geq t_1(B)$, then there exists $M > 0$ such that

$$|u_t|_r^r + \|v\|^2 + \|u\|^2 \leq M, \quad (3.23)$$

which completes the proof of the desired results.

Corollary 3.5. *For any $B \subset E_0$, there exists $T = T(B)$, such that*

$$\int_t^{+\infty}(\|u_t(\tau)\|^2 + \|u_{tt}(\tau)\|^2)d\tau + \|u_t\|^2 + \|u_{tt}\|^2 \leq M_2, \quad (3.24)$$

provided that $t \geq T$, where M_2 is a positive constant only depends on C (given by (3.8) *and* (3.9)) *and M (given by* (3.23)), Ω.

Proof. Multiplying (1.1) by $u_t(t)$, u_{tt}, respectively integrating in dx over Ω, and then integrating in $d\tau$ on $[t, +\infty]$ and Theorem 3.4 and Sobolev embedding $H_0^1 \hookrightarrow L^3$, we can conclude above.

Hereafter, we always assume that for some $\delta > 0$, $v = u_t + \delta u$ and

$$B_0 = \{(u, u_t) \in E_0; \|u\| + \|u_t\| \leq M\},$$

is the bounded absorbing set of $\{S(t)\}_{t\geq 0}$ in E_0 obtained in Theorem 3.4.

3.2. Global attractor

Let $u(x, t)$ be a unique weak solution of (1.1) corresponding to the initial data $z_0 = (u_0, u_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$. We decompose u into the sum

$$u(t) = v(t) + w(t),$$

where $v(t)$ and $w(t)$ are the solutions to the problems

$$\begin{cases} (|v_t|v_t)_t - \Delta v - \mu\Delta v_t - \Delta v_{tt} = f_0(v), \\ v|_{\partial\Omega} = 0, \\ v(0) = u_0,\ v_t(0) = u_1 \end{cases} \tag{3.25}$$

and

$$\begin{cases} (|u_t|u_t - |v_t|v_t)_t - \Delta w - \mu\Delta w_t - \Delta w_{tt} = f(u) - f_0(v), \\ w|_{\partial\Omega} = 0, \\ w(0) = 0,\ w_t(0) = 0. \end{cases} \tag{3.26}$$

It is convenient to denote

$$z(t) = (u(t), u_t(t)), \quad z_d(t) = (v(t), v_t(t)), \quad z_e(t) = (w(t), w_t(t)).$$

Lemma 3.6. *For any $R \geq 0$, there exist $M_0 = M_0(R) \geq 0$ and $k_0 = k_0(R) > 0$, such that whenever $\|z_0\|_0 \leq R$, it follows that*

$$\|z_d(t)\|_0 \leq M_0 e^{-k_0 t}, \quad \forall t \in \mathbb{R}^+,$$

the constants M_0 and k_0 only depend on R.

Proof. Denoting $\xi = v_t + \varepsilon v$, then rewrite the equation of (3.25) as follows:

$$(|v_t|v_t)_t - (1 - \mu\varepsilon + \varepsilon^2)\Delta v - \Delta\xi_t - (\mu - \varepsilon)\Delta\xi = f_0(v), \tag{3.27}$$

and set

$$E(t) = (1 - \mu\varepsilon + \varepsilon^2)\|v\|^2 + \|\xi\|^2 - 2F_0(v). \tag{3.28}$$

Associating to (3.5) we get the differential inequality

$$\frac{d}{dt}E(t) + 2\varepsilon(1 - \mu\varepsilon + \varepsilon^2)\|v\|^2 + 2(\mu - \varepsilon)\|\xi\|^2$$

$$\leq -4\int_\Omega |\xi - \varepsilon v| v_{tt}\xi$$

$$\leq 4\|v_{tt}\|\|\xi\|^2 + 4\varepsilon\|v_{tt}\|\|\xi\|\|v\|$$

$$\leq (\mu - \varepsilon)\|\xi\|^2 + \varepsilon(1 - \mu\varepsilon + \varepsilon^2)\|v\|^2 + C\|v_{tt}\|^2(\|\xi\|^2 + \|v\|^2). \tag{3.29}$$

Multiplying (3.25) by $v_t(t)$, integrating in dx over Ω, then integrating in $d\tau$ on $[0, t]$, then

$$\sup_{\|z_0\|_0 \leq R}\ \sup_{t \in \mathbb{R}^+} \|z_d(t)\|_0 < +\infty.$$

So we get

$$-2F_0(v) \leq C(|v|^2 + |v|_6^6) \leq k\|v\|^2, \quad \forall t \geq 0,$$

where $k = k(R) \geq 1$. Let

$$w = w(\varepsilon, k) = \frac{1 - \mu\varepsilon + \varepsilon^2}{1 - \mu\varepsilon + \varepsilon^2 + k} \leq \frac{1}{k}, \quad \forall \varepsilon > 0.$$

Then by (3.28), there exists a constant w such that

$$w(E(t) - \|\xi\|^2) \leq (1 - \mu\varepsilon + \varepsilon^2)\|v\|^2. \tag{3.30}$$

Combining (3.28), (3.29), (3.30) and the Young's inequality, we are led to the differential inequality

$$\frac{d}{dt}E(t) + \varepsilon w E(t) + (\mu - (1 + w)\varepsilon)\|\xi\|^2 \leq C\|v_{tt}\|^2 E(t).$$

Let ε be small enough such that

$$\varepsilon \leq \frac{\mu}{2(1 + w)},$$

then

$$\mu - (1 + w)\varepsilon \geq 0.$$

By Lemma 2.1 and Corollary 3.5, so there holds

$$E(t) \le CE(0)e^{-k_0 t}.$$

Putting together (3.28), the proof is finished.

Lemma 3.7. *For any time* $T \in \mathbb{R}^+$ *and every* μ, *there exists a compact set* $\mathcal{K}_T \subset H_0^1(\Omega) \times H_0^1(\Omega)$ *such that*

$$\bigcup_{z_0 \in B_0} z_c(t) \in \mathcal{K}_T, \quad \forall t \in [0, T].$$

Proof. Associating to Theorem 3.4 and Lemma 3.6, there exists a positive constant C, such that

$$|u_t|_3^3 + |v_t|_3^3 + \|u_{tt}\|^2 + \|v_{tt}\|^2 + \|u\|^2 + \|v\|^2 \le C, \quad \forall t \in \mathbb{R}^+. \tag{3.31}$$

Setting

$$\sigma = \min\left\{\frac{1}{4}, \frac{5-p}{2}\right\},$$

multiplying the equation (3.26) by $A^\sigma w_t$, we have

$$\frac{1}{2}\frac{d}{dt}\|z_c\|_\sigma^2 + \mu|A^{(1+\sigma)/2}w_t|^2 + \langle(|u_t|u_t - |v_t|v_t)_t, A^\sigma w_t\rangle$$

$$= \langle f(u) - f(v), A^\sigma w_t\rangle + \langle f_1(v), A^\sigma w_t\rangle, \tag{3.32}$$

where

$$\|z_c\|_\sigma^2 = |A^{(1+\sigma)/2}w|^2 + |A^{(1+\sigma)/2}w_t|^2.$$

By (3.3), we get

$$\begin{aligned}\langle f(u) - f(v), A^\sigma w_t\rangle &\le C\int_\Omega (1 + |u|^4 + |v|^4)|w||A^\sigma w_t| \\ &\le C(1 + \|u\|^4 + \|v\|^4)|A^{(1+\sigma)/2}w||A^{(1+\sigma)/2}w_t| \\ &\le C\|z_c\|_\sigma^2 + \frac{\mu}{3}|A^{(1+\sigma)/2}w_t|^2, \end{aligned}\tag{3.33}$$

combining with $p/(5-2\sigma) \le 1$ and (3.5) we have

$$\begin{aligned} \langle f_1(v),\, A^{\sigma} w_t \rangle &\le C \int_{\Omega} (1 + |v|^p) |A^{\sigma} w_t| \\ &\le C(1 + \|v\|^p) \|A^{(\sigma+1)/2} w_t\| \\ &\le C + \frac{\mu}{3} |A^{(\sigma+1)/2} w_t|^2. \end{aligned} \tag{3.34}$$

Combining with $1/(4-2\sigma) \le 1$ and (3.31) we have

$$\begin{aligned} &\langle (|u_t| u_t - |v_t| v_t)_t,\, A^{\sigma} w_t \rangle \\ &\le 2 \int_{\Omega} (|u_t||u_{tt}| + |v_t||v_{tt}|) |A^{\sigma} w_t| \\ &\le C(\|u_t\| \|u_{tt}\| + \|v_t\| \|v_{tt}\|) |A^{\sigma} w_t|_{\frac{6}{2\sigma+1}} \\ &\le C + \frac{\mu}{3} |A^{(\sigma+1)/2} w_t|^2. \end{aligned} \tag{3.35}$$

Putting together (3.33), (3.34) and (3.35) there holds

$$\frac{d}{dt} \|z_c\|_{\sigma}^2 \le C \|z_c\|_{\sigma}^2 + C,$$

by the Gronwall lemma, we find

$$\|z_c\|_{\sigma}^2 \le e^{Ct} - 1, \qquad \forall t \in [0, T].$$

The proof is finished.

Collecting now Theorem 3.3, Lemma 3.6 and Lemma 3.7, we establish that $\{S(t)\}_{t \ge 0}$ is asymptotically smooth. Therefore, by means of well-known results of the theory of dynamical systems we get

Theorem 3.8. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with smooth boundary, and we assume that* f *satisfies* (3.3)-(3.7), *the semigroup* $\{S(t)\}_{t \ge 0}$ *possesses a global attractor* $\mathscr{A}$ *on* $H_0^1(\Omega) \times H_0^1(\Omega)$.

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